Top-k Indexes Made Small and Sweet

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Small

Easy to implement.

Sweet

With non-trivial theoretical guarantees.

Computation Models

- RAM
- EM
  - Disk block size B
  - Running time: number of I/Os
  - As far as this talk is concerned, think about RAM as EM with $B = 2$.

"Enough Already" [Carey and Kossmann'97]

What to do if the database returns too many results?

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What to do if the database returns too many results?

Popular Approach: Top-k

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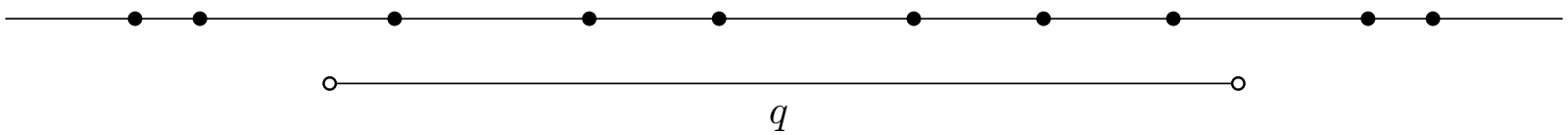
- Returns only the k elements with the highest priorities.
- Every reporting query has its top-k version.

Example 1: 1d Top-k Range Reporting

Range Reporting

"Find the hotels whose prices are between 100 and 200 dollars per night."



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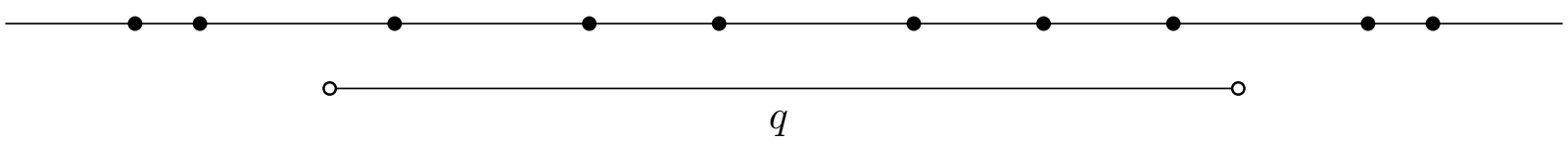
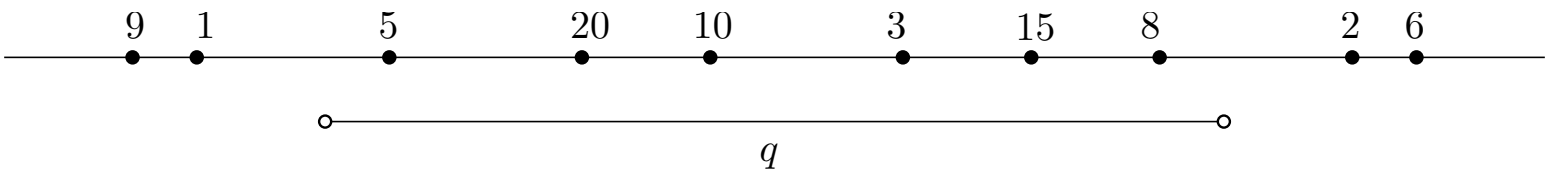


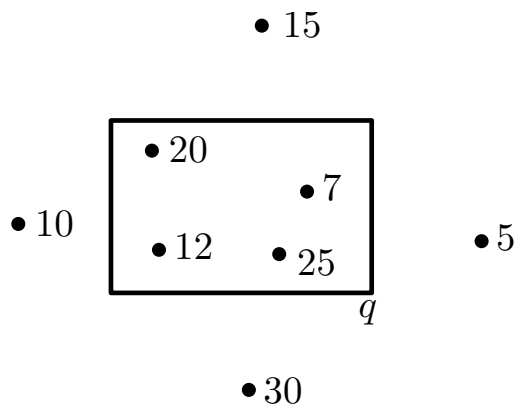
Top-k Range Reporting

"Find the k best-rated hotels whose prices are between 100 and 200 dollars per night."



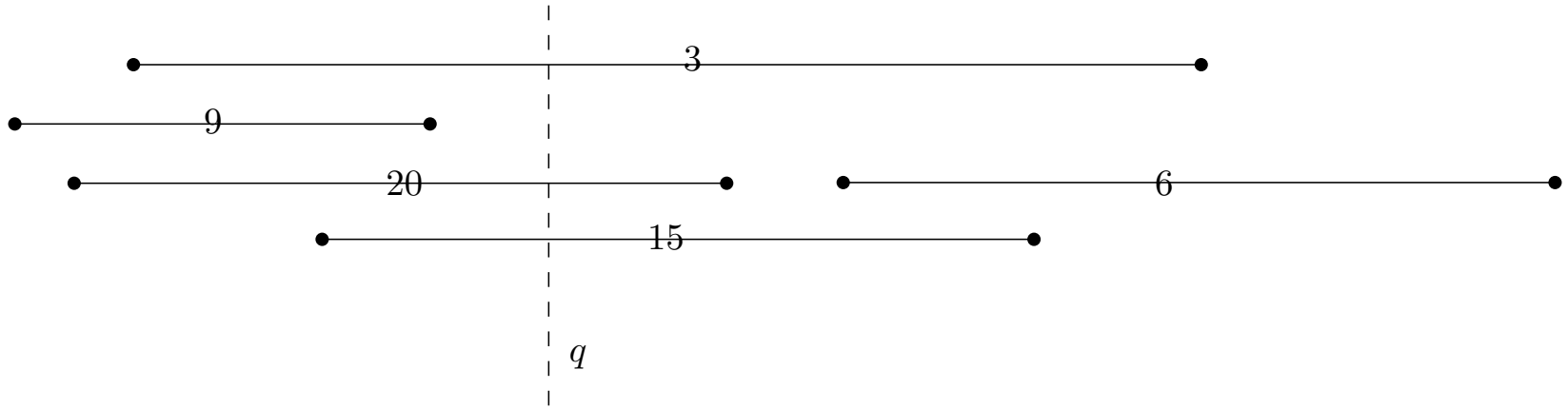
Example 2: 2d Top-k Range Reporting

"Find the k best-rated hotels in the Manhattan area".



Example 3: 1d Top-k Stabbing

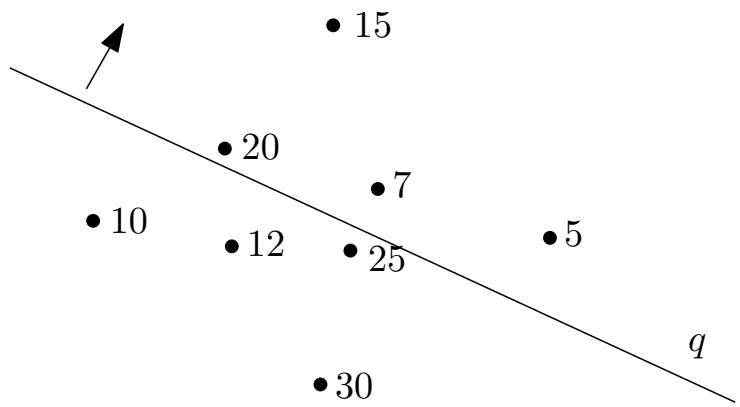
"Find the k accounts with the largest balances active on 1 Jan 2016."



Example 4: 2d Top-k Halfspace Reporting

"Find the k best-rated hotels satisfying

$$0.1 \cdot \text{distance} + 0.9 \cdot \text{price} \geq 5."$$



Top-k Literature

Small (system): A LOT OF work!

Top-k Literature

Small (system): A LOT OF work!

Sweet (theory):

- Serious attention only in the last few years.
- Best understood: 1d top-k range reporting.
- Recent efforts: more sophisticated top-k problems.
- All with sophisticated designs.

This Talk

Aims to significantly simplify the design of top-k structures with non-trivial theoretical guarantees for a large set of problems.

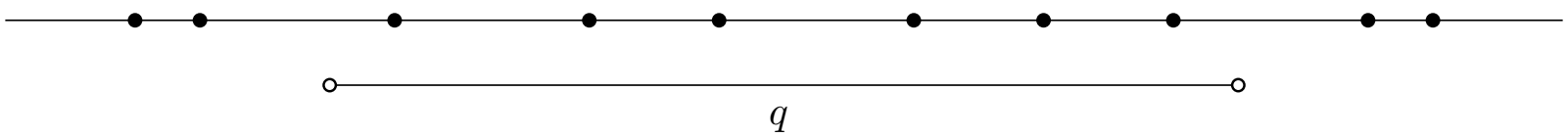
Our results immediately imply structures with excellent efficiency for all the aforementioned problems (and much more).

- No/little efforts required.

Results taken from our work in PODS'12, PODS'14, and PODS'16.

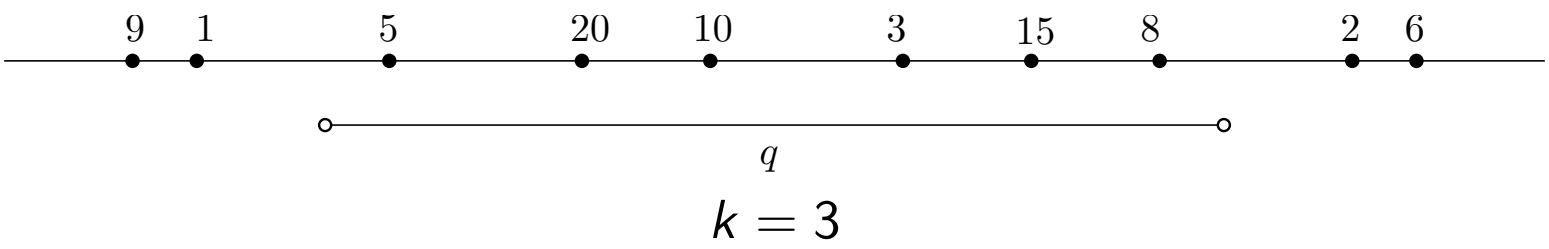
Reporting Queries in General

- Input D: a set of n elements from a domain $\mathbb{D}$.
- $\mathbb{Q}$: the set of all permissible predicates.
- A query
  - chooses a predicate $q \in \mathbb{Q}$
  - reports the set $q(D)$ of elements in D satisfying q.



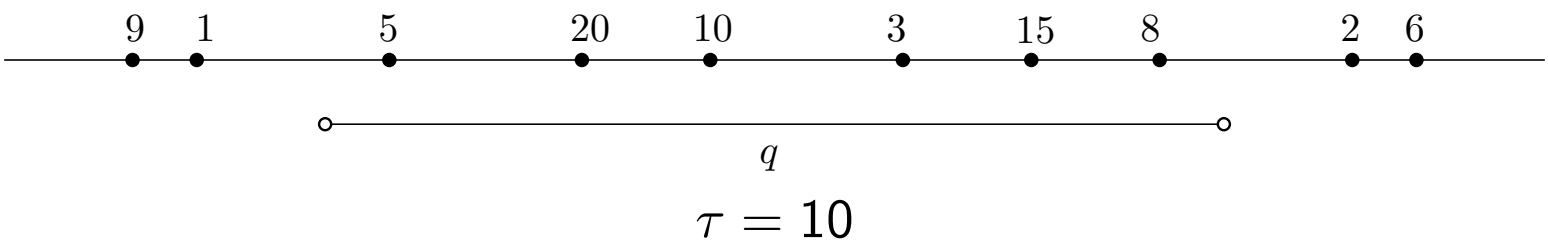
Top-k Reporting Queries in General

- Input D: a set of n elements from a domain $\mathbb{D}$.
  - Each $e \in D$ is associated with a real-valued weight $w(e)$.
- $\mathbb{Q}$: the set of all permissible predicates.
- A query
  - chooses a predicate $q \in \mathbb{Q}$ and a positive integer k
  - reports the k elements in $q(D)$ with the highest weights.



Prioritized Reporting Queries in General

- Input D: a set of n elements from a domain $\mathbb{D}$.
  - Each $e \in D$ is associated with a real valued weight $w(e)$.
- $\mathbb{Q}$: the set of all permissible predicates.
- A query
  - chooses a predicate $q \in \mathbb{Q}$ and a real-value τ
  - reports the elements in $q(D)$ whose weights are $\geq \tau$.



Common Pattern in All the Theoretical Top-k Solutions

1. Design a structure for prioritized reporting (often easy).
2. Convert a top-k query to a prioritized query (usually difficult).

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The Pattern is Compulsory!

Prioritized reporting can always be reduced to top-k reporting.

- If there is a top-k structure of $\mathcal{S}_{top}(n)$ space and $\mathcal{Q}_{top}(n) + O(k/B)$ query time
- Then there is a prioritized structure of $O(\mathcal{S}_{top}(n))$ space and $O(\mathcal{Q}_{top}(n)) + O(t/B)$ query time, where t is the number of elements reported.

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Implication: prioritized reporting no harder than top-k reporting.

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Suppose that there is a **prioritized structure** of $\mathcal{S}_{pri}(n)$ space and $\mathcal{Q}_{pri}(n) + O(t/B)$ query time.

We want to leverage the structure as a **BLACK BOX** to design a top-k structure of $\mathcal{S}_{top}(n)$ space and $\mathcal{Q}_{top}(n) + O(k/B)$ query time. How good can functions $\mathcal{S}_{top}(n)$ and $\mathcal{Q}_{top}(n)$ be?

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The Grand Wish

If one can prove $\mathcal{S}_{top}(n) = O(\mathcal{S}_{pri}(n))$ and $\mathcal{Q}_{top}(n) = O(\mathcal{Q}_{pri}(n))$, then it will mean that the two problems actually have the same hardness!

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Our Result 1

$$\mathcal{S}_{top}(n) = O(\mathcal{S}_{pri}(n)) \text{ and } \mathcal{Q}_{top}(n) = O(\mathcal{Q}_{pri}(n) \cdot \log_B n)$$

subject to very mild conditions.

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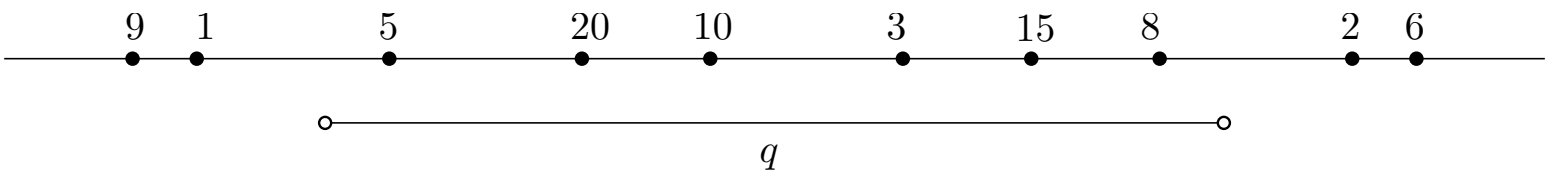
What if we want no deterioration at all?

Max Reporting Queries in General

- Input D: a set of n elements from a domain $\mathbb{D}$.
  - Each $e \in D$ is associated with a real-valued weight $w(e)$.
- $\mathbb{Q}$: the set of all possible predicates.
- A query
  - chooses a predicate $q \in \mathbb{Q}$
  - reports the element in $q(D)$ with the highest weight.



Obviously no harder than top-k reporting.

Suppose that there is

- A **prioritized structure** of $\mathcal{S}_{pri}(n)$ space that answers a query in $\mathcal{Q}_{pri}(n) + O(t/B)$ time, and can be updated in $\mathcal{U}_{pri}(n)$ time;
- A **max structure** of $\mathcal{S}_{max}(n)$ space that answers a query in $\mathcal{Q}_{max}(n)$ query time, and can be updated in $\mathcal{U}_{max}(n)$ time.

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Our Result 2

There is a top-k structure with space $\mathcal{S}_{top}(n)$, query time $\mathcal{Q}_{top}(n) + O(k/B)$, and update time $\mathcal{U}_{top}(n)$ time where

$$\begin{aligned} \mathcal{S}_{top}(n) &= O(\mathcal{S}_{pri}(n) + \mathcal{S}_{max}(n)) \text{ in expectation} \\ \mathcal{Q}_{top}(n) &= O(\mathcal{Q}_{pri}(n) + \mathcal{Q}_{max}(n)) \text{ in expectation} \\ \mathcal{U}_{top}(n) &= O(\mathcal{U}_{pri}(n) + \mathcal{U}_{max}(n)) \text{ in expectation} \end{aligned}$$

subject to mild conditions.

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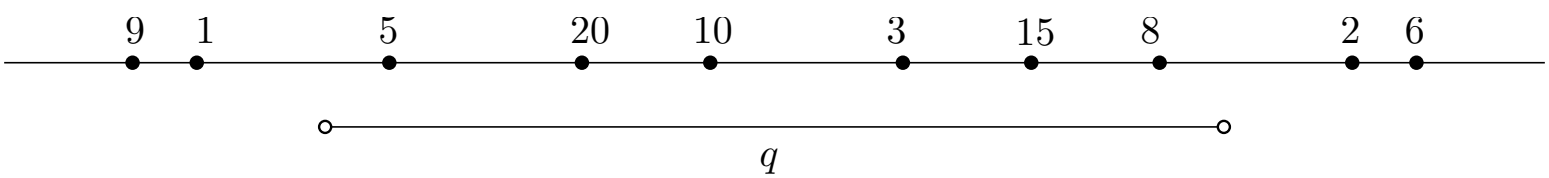
subject to mild conditions.

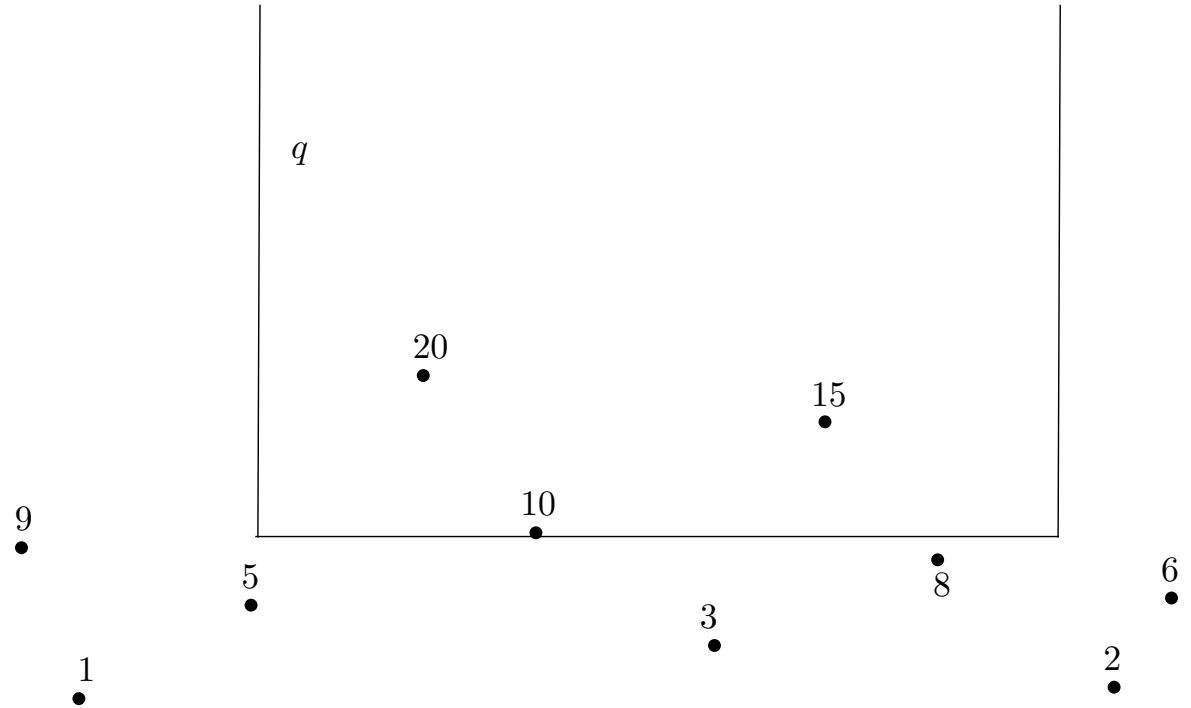
Implication: in terms of expected performance, top-k is as hard as solving prioritized reporting and max reporting simultaneously.

Application 1: 1d Top-k Range Reporting



Application 1: 1d Top-k Range Reporting

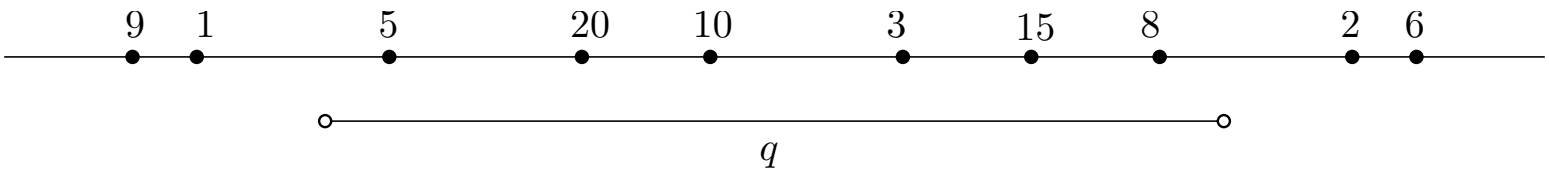
Prioritized reporting: 3-sided range reporting



External priority search tree [Arge et al.'99]:
$\mathcal{S}_{pri}(n) = O(n/B)$, $\mathcal{Q}_{pri}(n) = O(\log_B n)$, $\mathcal{U}_{pri}(n) = O(\log_B n)$

Application 1: 1d Top-k Range Reporting

Max reporting



B-tree:

$\mathcal{S}_{max}(n) = O(n/B)$, $\mathcal{Q}_{max}(n) = O(\log_B n)$, $\mathcal{U}_{max}(n) = O(\log_B n)$

Application 1: 1d Top-k Range Reporting

$\mathcal{S}_{pri}(n) = O(n/B)$, $\mathcal{Q}_{pri}(n) = O(\log_B n)$, $\mathcal{U}_{pri}(n) = O(\log_B n)$
$\mathcal{S}_{max}(n) = O(n/B)$, $\mathcal{Q}_{max}(n) = O(\log_B n)$, $\mathcal{U}_{max}(n) = O(\log_B n)$

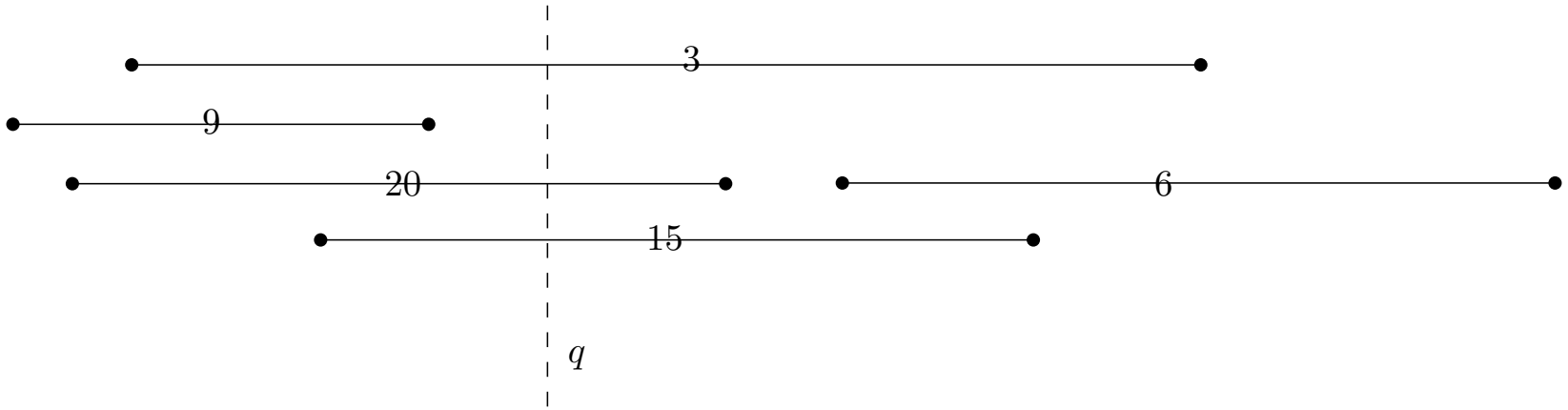
Our Result 2 immediately implies

$\mathcal{S}_{top}(n) = O(n/B)$, $\mathcal{Q}_{top}(n) = O(\log_B n)$, $\mathcal{U}_{top}(n) = O(\log_B n)$
all in expectation

Current state of the art [Tao'14, Brodal'15] (complicated structures!)

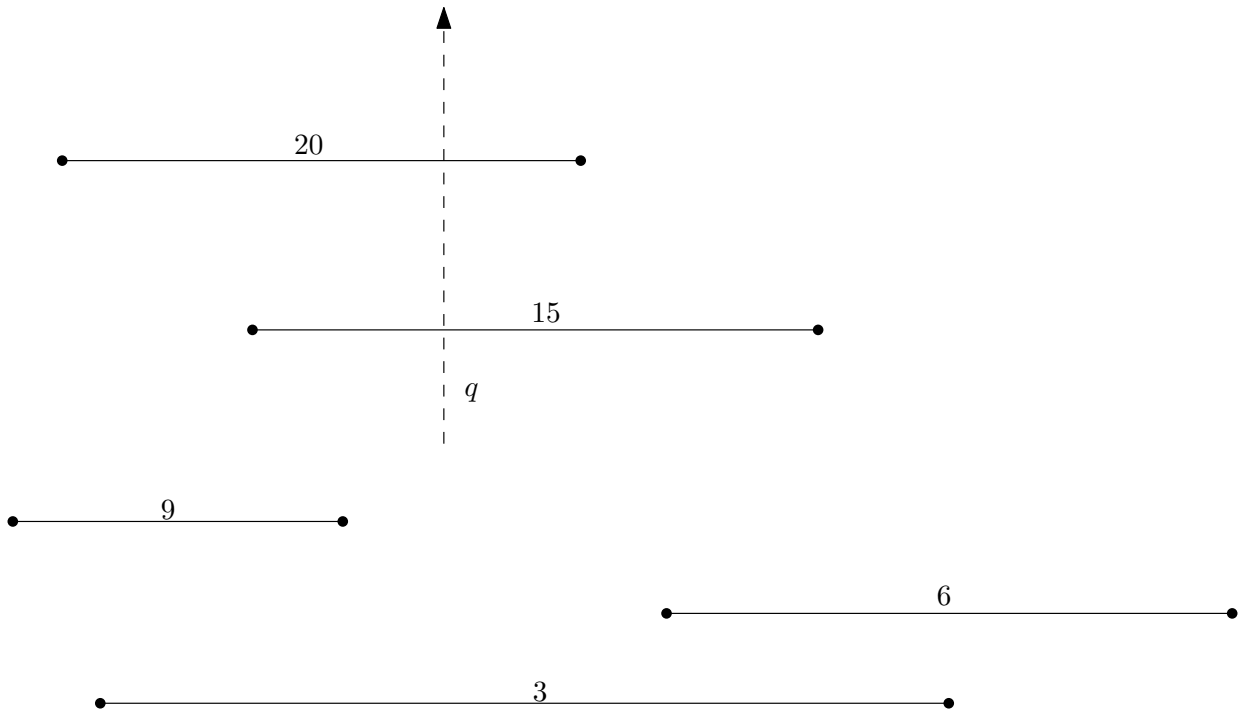
$\mathcal{S}_{top}(n) = O(n/B)$, $\mathcal{Q}_{top}(n) = O(\log_B n)$, $\mathcal{U}_{top}(n) = O(\log_B n)$ amortized

Application 2: 1d Top-k Stabbing



Application 2: 1d Top-k Stabbing

Prioritized reporting: Ray stabbing

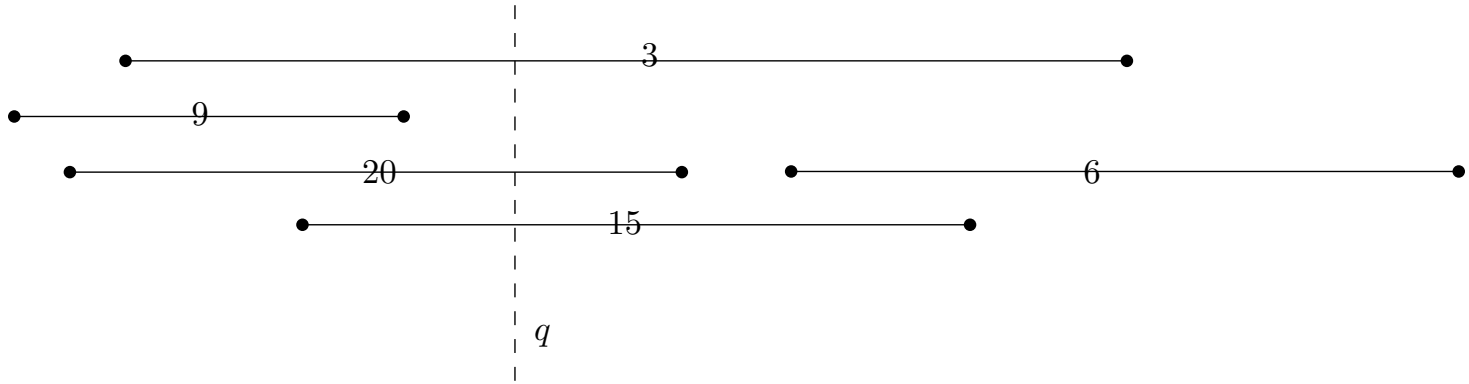


Ray stabbing structure [Tao'14]:

$\mathcal{S}_{pri}(n) = O(n/B)$, $\mathcal{Q}_{pri}(n) = O(\log_B n)$, $\mathcal{U}_{pri}(n) = O(\log_B n)$ amortized

Application 2: 1d Top-k Stabbing

Max reporting



Stabbing max structure [Agarwal et al.'12]:
$\mathcal{S}_{max}(n) = O(n/B)$, $\mathcal{Q}_{max}(n) = O(\log_B n)$, $\mathcal{U}_{max}(n) = O(\log_B n)$ amortized

Application 2: 1d Top-k Stabbing

$\mathcal{S}_{pri}(n) = O(n/B)$, $\mathcal{Q}_{pri}(n) = O(\log_B n)$, $\mathcal{U}_{pri}(n) = O(\log_B n)$ amortized
$\mathcal{S}_{max}(n) = O(n/B)$, $\mathcal{Q}_{max}(n) = O(\log_B n)$, $\mathcal{U}_{max}(n) = O(\log_B n)$ amortized

Our Result 1 immediately implies

$\mathcal{S}_{top}(n) = O(n/B)$, $\mathcal{Q}_{top}(n) = O(\log_B^2 n)$ worst case

Our Result 2 immediately implies

$\mathcal{S}_{top}(n) = O(n/B)$, $\mathcal{Q}_{top}(n) = O(\log_B n)$, $\mathcal{U}_{top}(n) = O(\log_B n)$ amortized all in expectation

Current state of the art

None!

A function $f(n)$ is **geometrically converging** if it satisfies two conditions:

- For any $n \geq B$:

$$\sum_{i=0}^{h} f\left(\frac{n}{c^i}\right) = O(f(n))$$

  for any value $c \geq 2$, where h is the largest integer i satisfying $n/c^i \geq B$.

- For any $n < B$, $f(n) = O(1)$.

Theorem 1.

Suppose that there is a prioritized structure of $\mathcal{S}_{pri}(n)$ space and query cost $\mathcal{Q}_{pri}(n) + O(t/B)$ such that $\mathcal{S}_{pri}(n)$ is geometrically converging, and

$$\mathcal{Q}_{pri}(n) \quad \geq \quad \log_B n.$$

Furthermore, suppose that the problem is polynomially bounded, namely, for any input D of n elements, there are only $n^{O(1)}$ distinct outcomes for $q(D)$ over all the possible predicates $q \in \mathbb{Q}$.

Then, there is a top-k structure of space $\mathcal{S}_{top}(n)$ and query time $\mathcal{Q}_{top}(n) + O(k/B)$ with

$$\mathcal{S}_{top}(n) = O(\mathcal{S}_{pri}(n))$$

$$\mathcal{Q}_{top}(n) = O\left(\mathcal{Q}_{pri}(n) \cdot \frac{\log n}{\log B + \log \frac{\mathcal{Q}_{pri}(n)}{\log_B n}}\right)$$

Theorem 2.

Suppose that there is

- A prioritized-reporting structure of $\mathcal{S}_{pri}(n)$ space that answers a query in $\mathcal{Q}_{pri}(n) + O(t/B)$ I/Os;
- A max-reporting structure of $\mathcal{S}_{max}(n)$ space that answers a query (i.e., $k = 1$) in $\mathcal{Q}_{max}(n)$ I/Os. It is required that
  - $\mathcal{S}_{max}(n) = O(n^2/B)$ for any $n \geq B$.
  - $\mathcal{S}_{max}(n)$ is geometrically converging.

Then, there is a top-k structure of expected space $\mathcal{S}_{top}(n)$ and expected query time $\mathcal{Q}_{top}(n) + O(k/B)$ with

$$\mathcal{S}_{top}(n) = O\left(\mathcal{S}_{pri}(n) + \mathcal{S}_{max}\left(\frac{6n}{B \cdot Q_{pri}(n)}\right)\right)$$

$$\mathcal{Q}_{top}(n) = O\left(\mathcal{Q}_{pri}(n) + \mathcal{Q}_{max}(n)\right).$$

Furthermore, if the prioritized and max structures support an update in $\mathcal{U}_{pri}(n)$ and $\mathcal{U}_{max}(n)$ I/Os respectively, then the top-k structure supports an update in $O(\mathcal{U}_{pri}(n) + \mathcal{U}_{max}(n))$ expected I/Os. If any of $\mathcal{U}_{pri}(n)$ and $\mathcal{U}_{max}(n)$ is amortized, so is the update cost of the top-k structure.

A problem is λ-**polynomially bounded** if for any input D of n elements, there are at most n^λ distinct outcomes for $q(D)$ over all $q \in \mathbb{Q}$, where λ is a constant.

Lemma 3 (Top-k Core-Set Lemma).

For any integer $K \geq 4\lambda \ln n$, there is a subset R of D such that

- $|R| \leq 12\lambda \cdot (n/K) \ln n$.
- *For any $q \in \mathbb{Q}$ satisfying $|q(D)| \geq 4K$, it holds that*
  - $|q(R)| > 8\lambda \ln n$
  - *The element with weight rank $\lceil 8\lambda \ln n \rceil$ in $q(R)$ has weight rank between K and $4K$ in $q(D)$.*

Foundation for our Result 1.

Lemma 4.

Let S be a set of n elements, and $K \geq 2$ a real value satisfying $n \geq 4K$. For a $(1/K)$-sample set R of S, the following hold simultaneously with probability at least 0.09:

- $|R| \geq 1$
- *The largest element in R has rank in S greater than K but at most $4K$.*

Foundation for our Result 2.

Reduction at a Very High Level

Structure

- Recursively sparsify the current dataset.
- Create a prioritized or max structure on each sparsified dataset.
- Create a prioritized structure on the original dataset.

Query

- Key is to handle small k (for worst case) or to guess the right τ (for expected case).
- Bottom up.

Open: is top-k reporting harder than prioritized reporting?

Our conjecture is no.

My Collaborators in the Top-k Project

Cheng Sheng

Rahul Saladi

THANK YOU!

Let Us Go Small and Sweet